# Solutions of higher-spin wave equations in external electromagnetic plane-wave fields



1976 J. Phys. A: Math. Gen. 9 149

(http://iopscience.iop.org/0305-4470/9/1/020)

J. Phys. A: Math. Gen., Vol. 9, No. 1, 1976. Printed in Great Britain. © 1976

# Solutions of higher-spin wave equations in external electromagnetic plane-wave fields

W Becker
Institut für Theoretische Physik der Universität Tübingen, 74 Tübingen, Germany

Received 30 May 1975, in final form 1 August 1975

**Abstract.** The Proca equation for a vector particle with an arbitrary magnetic dipole and electric quadrupole moment and the Rarita–Schwinger equation in the presence of an external electromagnetic plane-wave field are reduced to systems of ordinary differential equations which can be solved in special cases. The propagation behaviour of the solutions is investigated and found to exhibit in general the inconsistencies predicted by Velo and Zwanziger. In some cases there are in addition tachyonic modes. Characteristic differences appear for linear and circular polarization of the external field.

## 1. Introduction

Since the work of Velo and Zwanziger (1969a, b, 1971) it is known that wave equations describing particles with spin 1 or more are beset with inconsistencies which appear already on the classical level. The propagation velocity of the solutions may exceed the speed of light or the solutions do not describe wave propagation at all. According to Velo and Zwanziger these phenomena are expected to occur for spin 1 particles with an anomalous quadrupole moment and all particles with spin greater than 1. The results of Velo and Zwanziger are based on the method of characteristics which determines at every point the ray cone of wave propagation. It has been emphasized by Mathews and Seetharaman (1973) that there may be difficulties even if the ray cone determined in this way is inside the light cone: the limiting velocity obtained from the ray cone may actually turn out to be the minimum instead of the maximum velocity. In fact, this kind of tachyonic behaviour has been found for a vector particle with anomalous dipole moment in the presence of a constant magnetic field (Tsai and Yildiz 1971, Mathews 1974), where the ray cone coincides with the light cone.

Because of the complexity of higher-spin equations few explicit solutions are known and most investigations concentrated on constant external fields (see also Minkowski and Seiler 1971). Thus it may be of interest to have more explicit solutions, particularly for non-constant external fields. Especially one might look for tachyonic modes which cannot be detected by the method of characteristics. One could also examine to what extent a ray cone which exceeds at some points the light cone and lies inside at others, actually influences the wavefunction.

External plane-wave fields allow for a surprisingly simple solution of the corresponding Klein–Gordon and Dirac equations (Volkov 1935). An anomalous magnetic dipole moment can be included in the latter (Becker and Mitter 1974). The vector case with an anomalous magnetic moment has been treated similarly (Becker and Mitter

1975, to be referred to as I; the vector case with minimal coupling has also been solved in the Duffin–Kemmer formalism by Federov and Radyuk 1975). In the present paper we shall deal with the simplest cases which are relevant for the discussion of the inconsistencies, ie the Proca equation with an electric quadrupole coupling and the Rarita–Schwinger equation.

In § 2 the solution of I is extended to include the quadrupole coupling. The resulting system of ordinary differential equations can be solved for a monochromatic field with circular polarization (MCP) and for linear polarization, if the anomalous magnetic moment has the particular value $\mu = 1$. In § 3 the Rarita–Schwinger equation for circular as well as linear polarization is reduced to two systems of four ordinary differential equations, each, which can be solved for MCP and for a constant crossed field. In § 4 the propagation behaviour of the solutions is discussed. For MCP we recover all the results of Velo and Zwanziger and, in addition, for the vector meson with quadrupole coupling, some modes come out to be tachyonic. On the other hand, for linear polarization we show in one case and suspect in others that the limiting velocities are causal. For spin $\frac{3}{2}$, the results for a constant crossed field coincide with MCP.

## 2. The Proca equation

The Proca equation for a vector particle with an anomalous magnetic dipole moment and an additional electric quadrupole moment characterized by the dimensionless real constants $\mu$ and $g$, respectively, in an external electromagnetic field reads

$$(\kappa^2 - \pi^2)\psi_\mu + \pi_\nu \pi_\mu \psi^\nu + \mathrm{i}\epsilon\mu F_{\mu\nu}\psi^\nu - \frac{\epsilon g}{\kappa^2}(\pi^\rho Q_{\nu\rho\mu} + Q_{\mu\rho\nu}\pi^\rho)\psi^\nu = 0 \qquad (2.1)$$

where $\pi_\mu = \mathrm{i}\partial_\mu - \epsilon A_\mu$, $Q_{\mu\nu\rho} = \partial_\mu F_{\nu\rho}$ and we have already omitted terms quadratic in $Q_{\mu\nu\rho}$ which vanish in a laser field due to $F_{\mu\nu}F^{\mu\nu} = 0$. Equation (2.1) incorporates the subsidiary condition

$$\kappa^2 \pi_\mu \psi^\mu - \mathrm{i}\epsilon(1-\mu)\pi_\mu F^{\mu\rho}\psi_\rho - \frac{\epsilon g}{\kappa^2}\pi_\mu(\pi_\rho Q^{\nu\rho\mu} + Q^{\mu\rho\nu}\pi_\rho)\psi_\nu = 0. \qquad (2.2)$$

We specialize now to an external plane-wave field

$$A_\mu(x) = a e_{i\mu} A_i(\xi), \qquad \xi = kx,$$
$$k e_i = 0, \qquad e_i e_j = -\delta_{ij}, \qquad (i = 1, 2). \qquad (2.3)$$

Using an *ansatz* with a Volkov exponential

$$\psi_\mu(x|p) = \exp\left(-\mathrm{i}px + \frac{\mathrm{i}\epsilon}{2pk}\int^\xi \mathrm{d}\xi'(-2pA(\xi') + \epsilon A^2(\xi'))\right)\Psi_\mu(\xi) \qquad p^2 = \kappa^2 \qquad (2.4)$$

inserting (2.2) into (2.1) and introducing the quantities

$$D_3 = \kappa k_\mu \Psi^\mu, \qquad D_i = pk e_{i\mu}\Psi^\mu + (p_i - \epsilon a A_i)k_\mu \Psi^\mu \qquad (2.5)$$

we arrive at the simple system of equations

$$D_3' = \mathscr{F}_i D_i, \qquad D_i' = -\mathscr{F}_i^* D_3 \qquad (2.6)$$

where

$$\mathcal{F}_i = \gamma F_i - \mathrm{i}\delta F_i', \qquad F_i = A_i',$$

$$\gamma = \frac{\epsilon a}{2\kappa}(1-\mu), \qquad \delta = \frac{\epsilon agpk}{2\kappa^3} \tag{2.7}$$

and the prime denotes differentiation with respect to $\xi$. (2.6) is almost identical with the system obtained earlier (I(12)). The essential differences are that $\mathcal{F}_i$ is now complex and depends on $p$. The latter point will turn out to become crucial for the question of acausal propagation.

We find again that the positive quantity

$$K = \sum_{i=1}^{3} D_i D_i^* \tag{2.8}$$

is constant.

Special solutions of the system (2.6) can be obtained in the case of a monochromatic plane wave with circular polarization (MCP). Rewritten in terms of

$$\mathcal{A}_i = \gamma A_i - \mathrm{i}\delta F_i, \qquad X = \mathcal{A}_i D_i, \qquad Y = \mathcal{F}_i D_i \tag{2.9}$$

the system (2.6) has constant coefficients. The solutions are proportional to $\exp(\mathrm{i}\lambda\xi)$ where $\lambda$ is specified by

$$\lambda^3 - (1+\gamma^2+\delta^2)\lambda - 2\gamma\delta = 0 \tag{2.10}$$

with real zeros for all values of $\gamma$ and $\delta$. For a monochromatic linearly polarized (MLP) plane wave the system (2.6) is reminiscent of Mathieu's differential equation. Due to Floquet's theorem there are solutions of the form

$$D_i = \mathrm{e}^{\mathrm{i}\Lambda\xi} d_i(\xi)$$

with periodic $d$, $d_i(\xi + 2\pi) = d_i(\xi)$ and, as a consequence of (2.8), real $\lambda$. If $\gamma = 0$, ie $\mu = 1$, the system is immediately solved for arbitrary $F$ and yields ($F_1 = F$, $F_2 = 0$)

$$\begin{aligned} D_3 &= d_3\,\mathrm{e}^{-\mathrm{i}\delta F} + d_1\,\mathrm{e}^{\mathrm{i}\delta F} \\ D_1 &= d_3\,\mathrm{e}^{-\mathrm{i}\delta F} - d_1\,\mathrm{e}^{\mathrm{i}\delta F} \\ D_2 &= \text{constant}. \end{aligned} \tag{2.11}$$

In this case we have a periodic solution for periodic $F$. Analogy with Mathieu's equation suggests that this happens also for other particular values of $\gamma$ and $\delta$.

## 3. The Rarita–Schwinger equation

The Rarita–Schwinger equation implies the constraints (Velo and Zwanziger 1969a)

$$\gamma_\mu \psi^\mu = \frac{\epsilon}{3\kappa^2}\gamma_5\gamma_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\psi_\nu \tag{3.1}$$

$$\pi_\mu \psi^\mu = \frac{\epsilon}{3\kappa^2}(\not\pi + \tfrac{3}{2}\kappa)\gamma_5\gamma_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\psi_\nu . \tag{3.2}$$

When these constraints are resubstituted, the equation takes the following form, which is a true equation of motion

$$(\not{\pi}-\kappa)\psi_\mu-(\pi_\mu+\tfrac{1}{2}\kappa\gamma_\mu)\frac{\epsilon}{3\kappa^2}\gamma_5\gamma_\nu\epsilon^{\nu\rho\sigma\lambda}F_{\sigma\lambda}\psi_\rho\psi_\nu=0. \tag{3.3}$$

(3.3) has to be solved for the $v$ and $i$ components and the $u$ component† is then obtained from (3.2). The solution is most conveniently achieved by splitting off the Volkov exponential as above (equation (2.4), where $\Psi_\mu(\xi)$ now denotes a spinor-vector) and making use of the projection technique introduced earlier (Becker and Mitter 1974). The quantities $(i=1,2)$

$$N^{(i)}(\xi)=\gamma_v\left(\frac{1}{pv}(p_i-\epsilon aA_i)\Psi_v-\Psi_i\right) \tag{3.4}$$

are found to satisfy a system of first-order differential equations where the algebra is reduced to $\underline{1}$, $\gamma_i$, $\sigma=\mathrm{i}\gamma_1\gamma_2$. With an *ansatz*

$$N^{(i)}=(a^{(i)}+b_k^{(i)}\gamma_k+c^{(i)}\sigma)\gamma_v\psi_0 \tag{3.5}$$

where $\psi_0$ denotes a constant spinor, the system reads explicitly

$$\begin{aligned}
&a^{(i)\prime}-\tfrac{2}{3}\nu F_ib_j^{(j)}-\tfrac{1}{3}\nu F_jb_j^{(i)}-\alpha F_j^2(\epsilon_{ik}c^{(k)}-\mathrm{i}a^{(i)})=0\\
&c^{(i)\prime}-\tfrac{2}{3}\mathrm{i}\nu F_i\epsilon_{jk}b_k^{(j)}-\tfrac{1}{3}\mathrm{i}\nu F_j\epsilon_{jk}b_k^{(i)}-\alpha F_j^2(\epsilon_{ik}a^{(k)}-\mathrm{i}c^{(i)})=0\\
&b_k^{(i)\prime}+\tfrac{2}{3}\nu F_i(a^{(k)}-\mathrm{i}c^{(j)}\epsilon_{jk})+\tfrac{1}{3}\nu F_ka^{(i)}-\tfrac{1}{3}\mathrm{i}\nu F_j\epsilon_{jk}c^{(i)}+\mathrm{i}\alpha F_j^2(b_k^{(i)}+b_i^{(k)}-\delta_{ik}b_l^{(l)})=0
\end{aligned} \tag{3.6}$$

with

$$\nu=\frac{\epsilon a}{\kappa},\qquad \alpha=\frac{\epsilon^2a^2pk}{9\kappa^4}. \tag{3.7}$$

So we have eight scalar functions as is necessary for the description of spin $\frac{3}{2}$ particles. The remaining components of $\Psi_\mu(\xi)$ can be obtained from these by purely algebraic operations.

Both in the MCP case and for linear polarization the system (3.6) can be further reduced. In the MCP case, for

$$\begin{aligned}
&A_\pm=A_i(a^{(i)}\pm c^{(i)}), &&B_\pm=F_i(a^{(i)}\pm c^{(i)})\\
&C_\pm=A_i(F_k\pm\mathrm{i}A_k)b_k^{(i)}, &&D_\pm=F_i(F_k\pm\mathrm{i}A_k)b_k^{(i)}
\end{aligned}$$

we obtain two systems of four equations, each, with constant coefficients. With an

† We use the following notation for the light-like components of a vector $a_\mu$:

$$a_u=\hat{n}_\mu a^\mu,\qquad a_v=n_\mu a^\mu,\qquad a_i=-e_{i\mu}a^\mu$$

where $n_\mu$, $\hat{n}_\mu$, $e_{i\mu}$ are defined by the plane-wave field (2.3)

$$n^\mu=\frac{1}{\omega\sqrt{2}}k^\mu=\frac{1}{\sqrt{2}}(1,\boldsymbol{n}),\qquad \hat{n}_\mu=\frac{1}{\sqrt{2}}(1,-\boldsymbol{n}).$$

ansatz $\exp[i(\mu-\alpha)\xi]$ the characteristic polynomial becomes

$$\mu^4 \mp 2\mu^3 - \mu^2(2\beta^2 - 1 + \tfrac{10}{9}\nu^2) \pm \mu(2\beta + \tfrac{10}{9}\nu^2) + \beta^4 - \beta^2(1 + \tfrac{2}{9}\nu^2) \mp \tfrac{4}{9}\nu^2\beta + \tfrac{1}{9}\nu^4 = 0, \quad \beta = \alpha \mp 1 \tag{3.8}$$

the zeros of which are always real. For linear polarization ($F_1 = F, F_2 = 0$) the reduction occurs for the combinations

$$\begin{aligned} &A_1 = \frac{1}{\sqrt{3}}(a^{(1)} - ic^{(2)}), \qquad A_2 = a^{(1)} + ic^{(2)} \\ &A_3 = b_i^{(i)}, \qquad A_4 = \frac{1}{\sqrt{3}}(b_1^{(1)} - b_2^{(2)}) \\ &A_1' - (\nu/\sqrt{3})FA_3 = 0 \\ &A_2' - (2\nu/\sqrt{3})FA_3 - (\nu/\sqrt{3})FA_4 + 2i\alpha F^2 A_2 = 0 \\ &A_3' + (\nu/\sqrt{3})FA_1 + \tfrac{2}{3}\nu F A_2 = 0 \\ &A_4' + (\nu/\sqrt{3})FA_2 + 2i\alpha F^2 A_4 = 0. \end{aligned} \tag{3.9}$$

The other system obtained for

$$\begin{aligned} &\bar{A}_1 = -(1/\sqrt{3})(a^{(2)} + ic^{(1)}), \qquad \bar{A}_2 = a^{(2)} - ic^{(1)} \\ &\bar{A}_3 = \epsilon_{ij} b_j^{(i)}, \qquad \bar{A}_4 = (1/\sqrt{3})(b_2^{(1)} + b_1^{(2)}) \end{aligned}$$

agrees with (3.9). Equations (3.9) imply a conservation law

$$\sum_{i=1}^{4} A_i A_i^* = \text{constant} \tag{3.10}$$

with the consequence that, as in the MLP vector case, solutions of the form $A_i \sim \exp(i\lambda\xi) a_i(\xi)$ with periodic $a_i$ have real $\lambda$.

For the constant crossed (CC) field case ($F = 1$) we have $A_i \sim \exp(i\lambda\xi)$ with $\lambda$ given by

$$\lambda^4 + 4\alpha\lambda^3 + \lambda^2(4\alpha^2 - \tfrac{10}{9}\nu^2) - \tfrac{20}{9}\nu^2\alpha\lambda + \tfrac{1}{3}\nu^2(\tfrac{1}{3}\nu^2 - 4\alpha^2) = 0 \tag{3.11}$$

with four real solutions for any values of $\alpha$ and $\nu$.

## 4. Propagation behaviour of the solutions

The law of dispersion which governs the propagation of the particles, ie the relation between the components of momentum replacing $p^2 = \kappa^2$ in the external field, is obtained by Fourier transforming the wavefunctions (2.4) and (3.4), respectively

$$\psi_\mu(p|p') = \int d^4x \exp(ipx)\psi_\mu(x|p'), \qquad p'^2 = \kappa^2. \tag{4.1}$$

If the external field is monochromatic this can be done explicitly by expanding the field-dependent part of the exponential according to

$$\exp\left(\frac{i\epsilon}{2pk}\int^{\xi} d\xi'(-2pA(\xi')+\epsilon A^2(\xi'))\right)$$

$$=\begin{cases} \exp\left(-i\frac{\epsilon^2 a^2}{2pk}\xi\right)\sum_{m=-\infty}^{\infty} J_m(\zeta)\exp[im(\xi+r)] \\ \text{for MCP } A_1=\cos\xi,\ A_2=-\sin\xi, \text{ where } \zeta e^{ir}=(\epsilon a/pk)(p_1+ip_2), \\ \exp\left(-\frac{i\epsilon^2 a^2}{4pk}\xi\right)\sum_{m=-\infty}^{\infty} e^{im\xi}\sum_{l=-\infty}^{\infty}(-1)^l J_l\left(\frac{\epsilon^2 a^2}{8pk}\right)J_{m-2l}\left(\frac{\epsilon a p_1}{pk}\right) \\ \text{for MLP } A_1=\cos\xi,\ A_2=0, \\ \frac{z}{\pi}\int_{-\infty}^{\infty} dm\, e^{im\xi}\exp\left[\frac{ip_1}{\epsilon a}\left(\frac{p_1^2}{3pk}-m\right)\right]\phi\left(z\left(m-\frac{p_1^2}{2pk}\right)\right) \\ \text{for CC } A_1=\xi,\ A_2=0, \text{ where } z=(2pk/\epsilon^2 a^2)^{1/3} \text{ and } \phi \text{ is the Airy} \\ \text{function in the notation of Ritus (1972).} \end{cases} \quad (4.2)$$

All solutions obtained so far for the remaining part $\Psi_\mu(\xi)$ are sums of terms proportional to

$$e^{i\lambda\xi}d(\xi)=\Sigma d_n\, e^{i(\lambda+n)\xi} \quad (4.3)$$

where the periodic part has been expanded in a Fourier series. In all cases we obtain for the law of dispersion

$$(p_{m+n}+\lambda k)^2=\kappa_*^2=\begin{cases}\kappa^2(1+\nu^2) & \text{MCP}\\ \kappa^2(1+\frac{1}{2}\nu^2) & \text{MLP}\\ \kappa^2 & \text{CC.}\end{cases} \quad (4.4)$$

Here $p_{m+n}=p+(m+n)k$ denotes the Zeldovich (1967) quasi-momentum. Obviously, all quasi-momenta fulfil the same law of dispersion and therefore from now on the index on $p$ will be omitted. In all cases of special interest we will have

$$\lambda=\lambda_0+\Gamma\frac{pk}{2\omega^2} \quad (4.5)$$

where $\lambda_0$ is independent of $p$ and may be absorbed in the definition of the quasi-momentum.

The group velocity now turns out be

$$\boldsymbol{v}(\boldsymbol{p})=\nabla_{\boldsymbol{p}}p_0=(1+\Gamma)^{-1}\left(\Gamma\boldsymbol{n}\pm\frac{\boldsymbol{p}(1+\Gamma)-\Gamma\boldsymbol{n}(\boldsymbol{np})}{[(1+\Gamma)(\boldsymbol{p}^2+\kappa_*^2)-\Gamma(\boldsymbol{np})^2]^{1/2}}\right) \quad (4.6)$$

with $\boldsymbol{n}$ a unit vector in the direction of $\boldsymbol{k}$, ie $k_\mu=\omega(1,\boldsymbol{n})$. In the directions parallel or antiparallel and transverse to the wavevector $\boldsymbol{n}$, equation (4.6) reduces to (up to terms of the order of $1/p^2$)†

$$v(p)=\begin{cases}\dfrac{n\Gamma}{1+\Gamma}\pm\dfrac{p}{p(1+\Gamma)}\left(1-\dfrac{(1+\Gamma)\kappa_*^2}{2p^2}\right) & \text{for } p\sim n\\[2ex] \dfrac{n\Gamma}{1+\Gamma}\pm\dfrac{p}{p\surd(1+\Gamma)}\left(1-\dfrac{\kappa_*^2}{2p^2}\right) & \text{for } pn=0\end{cases} \tag{4.7}$$

where $p=|p|$. From (4.7) the limiting velocities $v_\infty$ can be read off while the relative sign of the next-to-leading term determines whether the modes are normal or tachyonic. The results are summarized in the following table:

| | $p\sim n$ | $pn=0$ |
|---|---|---|
| $\Gamma>1$ | $v_\infty=1$ or $v_\infty<1$ (tachyonic) | $v_\infty<1$ |
| $1>\Gamma>0$ | $v_\infty=1$ or $v_\infty<1$ | $v_\infty<1$ |
| $0>\Gamma>-1$ | $v_\infty=1$ or $v_\infty>1$ | $v_\infty>1$ |
| $\Gamma<-1$ | $v_\infty=1$ (tachyonic) or $v_\infty>1$ | no propagation |

The two values in the first column correspond to the different signs in equation (4.7). In the transverse direction in both cases the same absolute value of the limiting velocity obtains. For $\Gamma=1(\Gamma=-1)$ the limiting velocities vanish (become infinite) in particular directions.

The discussion of the propagation behaviour of the spin 1 and $\frac{3}{2}$ particles now amounts to writing down the respective values of $\Gamma$. We have tacitly assumed that $\Gamma$ is real. In fact, all characteristic exponents corresponding to the solutions obtained in §§ 2 and 3 fulfil this requirement, which is necessary for hyperbolicity. The vector meson without quadrupole moment behaves perfectly causal: in the MCP case the characteristic exponents $\lambda=0,\pm\surd(1+\gamma^2)$ which are the solutions of (2.10) do not depend on $pk$ and the solutions for MLP are periodic, ie $\Gamma=0$ in both cases. For non-vanishing quadrupole moment the case $\gamma=0$, ie $\mu=1$, is distinguished by its simplicity. For MCP we have $\lambda=0,\pm\surd(1+\delta^2)$ which fits the *ansatz* (4.5) for either small or large $\delta$. In the latter case

$$\Gamma=\pm\nu g\left(\frac{\omega}{\kappa}\right)^2. \tag{4.8}$$

This is the limit, even for $\gamma\neq 0$, if $pk$ becomes large. So we have all kinds of peculiarities as discussed above. The loss of hyperbolicity which occurs if $\Gamma<-1$ agrees with a general theorem due to Velo (1975). This theorem states hyperbolicity of a system which is equivalent to (2.1) under some conditions which reduce in the plane-wave field case to

$$\kappa^2-\frac{\epsilon g^2}{\kappa^4}\sum_{i<j}(Q_{0ij})^2\geq\epsilon_0>0.$$

This yields $\Gamma^2<1$ as a sufficient condition for hyperbolicity. The fact that $\Gamma$ (equation (4.8)) is linear with respect to $\nu$ in connection with the preceding table reflects an

† It does not make much sense to discuss the whole dependence on $p$ since in general (4.5) will turn out to be valid only for large $\lambda$ which in turn generally means large $p$.

observation made by Jenkins (1973b) for spin 1 theories: if the limiting velocity stays below the speed of light for some value of the coupling constant it exceeds it for the opposite value.

On the other hand for MLP and $\gamma = 0$ we have the periodic solutions (2.11) which do not show any inconsistencies for arbitrary values of $\Gamma$. The solutions are safely non-periodic as soon as $\gamma \neq 0$. In the limit of large $pk$, however, if $\gamma$ can be neglected in comparison to $\delta$, we expect the solutions to approach again (2.11). This would mean that the limiting velocities agree with the speed of light. Tachyonic modes will probably exist.

For spin $\frac{3}{2}$ particles and MCP we have in the limit of large $\alpha$ two double zeros of (3.8), so that $\lambda = 0, -2\alpha$. We recover completely the results of Velo and Zwanziger (1969a): an ordinary ray ($\lambda = 0$) with a causal and an extraordinary ray

$$\lambda = -2\alpha \qquad \Gamma = -\left(\frac{2\nu\omega}{3\kappa}\right)^2 \tag{4.9}$$

with an acausal limiting velocity as well as loss of hyperbolicity if $\Gamma < -1$. Note the definite sign of $\Gamma$ in contrast to equation (4.8). The extraordinary ray exhibits tachyonic behaviour in either of the parallel directions.

For MLP we have no explicit solutions of the system (3.9). Generally, the solutions will be non-periodic. As in the vector MLP case, however, if in the limit of very large $pk$ the terms proportional to $\nu$ can be neglected, we have the simple periodic solutions $A_{1,2} = \text{constant}$, $A_{2,4} \sim \exp(-2i\alpha \int F^2)$. This argument, if correct, would indicate a causal limiting velocity but not exclude tachyonic modes.

For the constant crossed field the results agree with those for MCP. For large $\alpha$, equation (3.11) has the double zeros $\lambda = 0, -2\alpha$, so that $\Gamma$ is given by (4.9) as before.

## 5. Conclusions

For a vector meson with quadrupole coupling and a spin $\frac{3}{2}$ particle in an external MCP field we have recovered all the results of Velo and Zwanziger: existence of ordinary rays and extraordinary rays which exhibit acausal propagation. If the external field exceeds some critical value some modes fail to propagate in the transverse direction and/or become tachyonic with a causal limiting velocity in the longitudinal directions.

In the case of linear polarization the results are quite different. In particular we should compare the results for the vector case with $\gamma = 0$, where we have an explicit solution for both polarizations. Whereas for circular polarization the complete catalogue of inconsistencies is observed, no vestiges of the locally acausal structure of the ray cones can be seen for linear polarization. Obviously the structure has been averaged out in calculating the group velocity. This does not necessarily mean that acausal behaviour cannot be observed even in this case: one would have to use wave packets which are localized to approximately the region of one wavelength of the plane-wave field in their frame of reference. For such wave packets the concept of group velocity becomes meaningless. Since the wave packets would decay very rapidly it is not clear, however, if acausal behaviour can be observed even in principle. The reason for the different behaviour in circularly and linearly polarized fields appears to be that the circularly polarized field resembles more a constant field than the linearly polarized one, since its absolute value is constant. Hence its influence cannot be as easily averaged out and the results agree, in fact, with those of the constant crossed field in the case of spin $\frac{3}{2}$.

The apparent absence of inconsistencies for linear polarization is also interesting with respect to an equivalence shown by Jenkins (1973a, b) for the vector case: the ray cones of the classical field equation agree with the light cone if and only if the $S$ operator in the quantized theory is Lorentz invariant, ie normal independent. The fact that the locally acausal structure of the ray cones is not reflected in the group velocity, raises the question to what extent the resulting non-covariance of the $S$ operator can be observed in $S$ matrix elements.

It should be noted, finally, that a further difficulty, namely a non-covariant loss of constraints in a particular frame of reference (Jenkins 1974), does not show up in the case of an external plane-wave field.

## Acknowledgments

I am grateful to Professor H Mitter for reading the manuscript.

## References

Becker W and Mitter H 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 1266–73
—1975 *Acta Phys. Austr.* to be published
Fedorov F I and Radyuk A F 1975 *Sov. J. Nucl. Phys.* **20** 274–6
Jenkins J D 1973a *Nuovo Cim. Lett.* **7** 559–62
—1973b *J. Phys. A: Math., Nucl. Gen.* **6** 1935–42
—1974 *J. Phys. A: Math., Nucl. Gen.* **7** 1129–34
Mathews P M 1974 *Phys. Rev.* D **9** 365–9
Mathews P M and Seetharaman M 1973 *Phys. Rev.* D **8** 1815–6
Minkowski P and Seiler R 1971 *Phys. Rev.* D **4** 359–66
Ritus V I 1972 *Ann. Phys., NY* **69** 555–82
Tsai Wu Yang and Yildiz A 1971 *Phys. Rev.* D **4** 3643–8
Velo G 1975 *Comm. Math. Phys.* **43** 171–80
Velo G and Zwanziger D 1969a *Phys. Rev.* **186** 1337–41
—1969b *Phys. Rev.* **188** 2218–22
—1971 *Tracts in Mathematics and Natural Sciences*, vol 4 ed A S Wightman (New York: Gordon and Breach)
Volkov D M 1935 *Z. Phys.* **94** 250–60
Zeldovich Ya B 1967 *Sov. Phys.-JETP* **24** 1006–8